Proof of the existence and uniqueness of a solution for the Haissinski equation with a capacitive wake function

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The existence and uniqueness of a solution for the Haissinski equation with a capacitive wake function has been analytically proved.

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The existence and uniqueness of a solution for the Haissinski equation is an interesting and important question, because the solution of this equation is the basis of the theory of longitudinal instabilities. The stability condition of an electron beam is studied by adding a small perturbation to this solution and finding the imaginary part of the frequency of modes of the electron beam [1–3]. It is known that the instability thresholds obtained with this method are in good agreement with the particle-tracking results [2,3]. Nonetheless, the properties of the solution of the Haissinski equation has not been studied sufficiently and analytically. This is because it is such a complicated nonlinear equation.

This means that there is possibility that this equation has either no solutions or many solutions. Indeed, it has been considered that a solution cannot exist for the inductive wake case beyond the threshold. Recently, it has been numerically proved that there is a solution for this case [4,5]. There seems to have always been a solution for arbitrary wake functions in previous numerical studies. It is desirable to show the existence and uniqueness of the solution for the Haissinski equation analytically; however, there is no complete successful analytical approach to prove them, except for a simple resistive wake case [6]. Here, we present a proof of the existence and uniqueness of a solution for a capacitive wake function.

Haissinski formulated a theory for bunch lengthening before mode-coupling instabilities occur [1]. Electrons in an accelerator interact with their environment because they are enclosed in metals (vacuum pipe, the rf cavities, etc.). This effect is represented by the wake function [6]. The wake field acting on an electron is determined by the distribution of electrons ahead of it. At the same time, the distribution of electrons is influenced by the wake field. Hence, to determine the distribution function, one should solve coupled nonlinear equations. The single-particle equations of motion are as follows:

$$\frac{d}{ds}\xi = -\frac{\omega_s}{c\sigma_\epsilon}\epsilon, \tag{1}$$

$$\frac{d}{ds}\epsilon = \frac{\omega_s\sigma_\epsilon}{c}\xi - \frac{e^2LN}{T_0E_0c}\int_\xi^\infty d\xi'\rho(\xi')W(\xi'-\xi). \tag{2}$$

Here, we have used the dimensionless parameter ξ as

$$\xi \equiv \frac{\omega_s}{\alpha\sigma_\epsilon}\tau, \tag{3}$$

where σ_ϵ is the nominal rms relative energy spread, c is the velocity of light, e is the electric charge of the electron, L is the total length of the pipe structure in which the wake field is generated, E_0 is the reference energy of the beam, N is the number of electrons in a bunch, T_0 is the revolution period of the beam, τ is the time displacement between an electron and the reference synchronous particle, ϵ is the relative energy $(E-E_0)/E_0$ with E being the electron energy, ω_s is the synchrotron oscillation frequency, α is the momentum compaction factor, and s is the longitudinal coordinate along the ring. The second term of Eq. (2) is the retarding force seen by a particle at ξ due to the longitudinal wake force that is produced by all particles in front of it; $\rho(\xi)$ is the particle density at location ξ.

In the presence of radiation, two more parameters are necessary: b, which is the damping coefficient and $D=b\sigma_\epsilon^2$, which is the diffusion coefficient representing the amount of quantum excitation due to photon emission. The dynamics with radiation may be described by the Fokker-Planck equation for the phase-space particle distribution $\phi(\epsilon,\xi,s)$ [7],

$$\frac{\partial\psi}{\partial s} = -\frac{\omega_s\epsilon}{\alpha\sigma_\epsilon}\frac{\partial}{\partial\xi}\psi + b\frac{\partial}{\partial\epsilon}\epsilon\psi + \left(\frac{\omega_s\sigma_\epsilon}{\alpha}\xi - \frac{e^2LN}{T_0E_0\sigma}\times\int_\xi^\infty d\xi'\rho(\xi')W(\xi'-\xi)\right)\frac{\partial}{\partial\epsilon}\psi + D\frac{\partial^2}{\partial\epsilon^2}\psi. \tag{4}$$

This equation has a static solution implicitly given by

$$\psi(\epsilon,\xi) = \exp\left(-\frac{\epsilon^2}{2\sigma_\epsilon^2}\right)\rho(\xi), \tag{5}$$

$$\rho(\xi) = A\exp\left(-\frac{\xi^2}{2} + \int_\xi^\infty d\xi' V(\xi')\right), \tag{6}$$

$$V(\xi) = \int_\xi^\infty d\xi'\rho(\xi')w(\xi'-\xi), \tag{7}$$

$$w(\xi'-\xi) = -\frac{e^2LN}{\omega_s\sigma_\epsilon T_0E_0}W(\xi'-\xi). \tag{8}$$

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Equation (6) is the Haissinski equation [1]. Here, A is the normalization constant: $\int \rho d\xi = 1$. Since $\rho(\xi)$ depends only on $\rho(\xi')$ for $\xi < \xi'$, and we know

$$\rho(\xi) \sim A \exp\left(-\frac{\xi^2}{2}\right), \quad \xi \to \infty, \tag{9}$$

Eq. (6) can be integrated from the head of the bunch to the tail for a given value of A. Let us call the result of such an integration $\rho(\xi;A)$ and define the "charge" Q as

$$Q = Q(A) = \int_{-\infty}^{\infty} \rho(\xi;A) d\xi. \tag{10}$$

If a value A exists such that $Q(A)=1$, it gives the solution of the Haissinski equation [2,3].

We usually find the solution of Eq. (6) numerically, because the wake force of vacuum chamber elements is very complicated. The wake function can often be parametrized by a sum of inductive, resistive, and capacitive wake functions [1,6,8]. Machines with deep cavities are slightly capacitive for a normal bunch, and become more capacitive for short bunches [9], i.e.,

$$w(\xi) = -C_0. \tag{11}$$

Here, we should notice that the sign of C_0 may be positive or negative, because the sign of C_0 depends on that of the momentum compaction factor.

Here, we analytically prove the existence and uniqueness of a solution of Eq. (6) with Eqs. (7) and (11). For the purpose of our proof, we show that "charge" $Q(A)$ is a monotonically increasing function of A, i.e.,

$$Q(0) = 0, \tag{12}$$

$$\frac{dQ(A)}{dA} > 0, \tag{13}$$

$$\text{there exists an } A_0 \text{ such that } \lim_{A \to A_0} Q(A) = \infty. \tag{14}$$

Equations (12), (13), and (14) assure the existence and uniqueness of the solution, because this causes the existence and uniqueness of A, which satisfies $Q(A)=1$. We mainly show Eqs. (13) and (14), because Eq. (12) is trivial.

Before our proof, it is necessary to know the relations that the solution of the Haissinski equation must satisfy. Equation (6) with Eqs. (7) and (11) is rewritten as

$$\frac{\rho'}{\rho} = -\xi + C_0 \int_{\xi}^{\infty} d\xi' \rho(\xi'), \tag{15}$$

where the prime denotes differentiation with respect to ξ. This may be solved as

$$\frac{\rho'}{\rho} = \pm\sqrt{2(-\log\rho - C_0\rho + D_0)}, \tag{16}$$

where D_0 is a constant of integration. By comparing Eq. (16) with Eq. (9), D_0 should be $\log A$. Further, since Eq. (9) means that ρ must be decreasing for a sufficiently large ξ, we have to choose the $-$ branch in Eq. (16) for $\xi \to \infty$.

If ρ' is always negative, there is no physical ρ, which means that $Q(A)$ is ill defined. We have to find the region of ξ where ρ' can be positive. According to Eq. (15), if there exists ξ_0, such that

$$-\xi_0 + C_0 \int_{\xi_0}^{\infty} d\xi' \rho(\xi') = 0, \tag{17}$$

ρ' can become positive for $\xi \leqslant \xi_0$. According to Eq. (16), this means that if there exists $\rho(\xi_0)$, such that

$$-\log\rho(\xi_0) - C_0\rho(\xi_0) + \log A = 0, \tag{18}$$

we may choose the $+$ branch in Eq. (16) for $\xi \leqslant \xi_0$. Following this manipulation, we may find a continuous solution.

Next, we should confirm whether the sign of ρ' really changes or not by using the original differential equation. If C_0 is positive, the right-hand side of Eq. (15) is monotonically increasing as ξ becomes smaller. Thus, the sign of ρ' becomes positive for $\xi \leqslant \xi_0$. On the other hand, if C_0 is negative, there occurs a change of the sign only for $\rho(\xi) \leqslant 1/|C_0|$. Otherwise, there is no "well-defined" ρ, even if ξ_0 exists. This condition for ρ may be rewritten to that for A by using Eq. (18). That is, A is arbitrary for a positive C_0, while $0 \leqslant A < 1/(|C_0|e)$ for a negative C_0, where $e = 2.7182, \ldots$. From the above discussions, we may also find that there is only one ξ_0, which means that ρ has only one relative maximum. Further, ρ cannot be larger than $1/|C_0|$ when C_0 is negative.

Thus, ρ must satisfy

$$\int_{\rho(\xi_\infty,A)}^{\rho(\xi,A)} \frac{d\rho}{\rho\sqrt{2(-\log\rho - C_0\rho + \log A)}} = -(\xi - \xi_\infty) \quad \text{for } \xi \geqslant \xi_0,$$

$$\int_{\rho(\xi_\infty,A)}^{\rho(\xi,A)} \frac{d\rho}{\rho\sqrt{2(-\log\rho - C_0\rho + \log A)}} = \xi - 2\xi_0(A) + \xi_\infty \quad \text{for } \xi \leqslant \xi_0, \tag{19}$$

where we use Eq. (16). The parameter ξ_∞ is an artificial point that goes to infinity as $\rho(\xi_\infty,A)$ goes to 0.

Before we proceed further, we should know the condition that the charge $Q(A)$ is well defined. According to Eqs. (10) and (19), $Q(A)$ is given as

$$Q(A) = \int_0^{\rho(\xi_0,A)} d\rho \sqrt{\frac{2}{-\log\rho - C_0\rho + \log A}}, \tag{20}$$

where we change the variable ξ to ρ. Here, we should notice that the integrand is divergent at $\rho = \rho(\xi_0)$, because $\rho(\xi_0)$ is given by Eq. (18). It is necessary to investigate whether $Q(A)$ converges or not.

Let us divide the integral region into the following form:

$$Q=\left[\int_0^{\rho(\xi_0)-\epsilon}+\int_{\rho(\xi_0)-\epsilon}^{\rho(\xi_0)}\right]d\rho\sqrt{\frac{2}{-\log\rho-C_0\rho+\log A}}, \tag{21}$$

where $0<\epsilon<\rho(\xi_0)$. The first term is obviously convergent. The second term, which we call Q_2, is given by

$$Q_2=\int_0^{\epsilon}dx\sqrt{\frac{2}{-\log\left[\rho(\xi_0)\left(1-\frac{x}{\rho(\xi_0)}\right)\right]-C_0[\rho(\xi_0)-x]+\log A}}, \tag{22}$$

where $x=\rho(\xi_0)-\rho$. Since $\log(1-x/\rho_0)$ may be expanded into a Taylor series for $0<x/\rho_0<1$, the following inequality may be obtained:

$$\begin{aligned}Q_2&=\int_0^{\epsilon}dx\sqrt{\frac{2}{\frac{x}{\rho(\xi_0)}+\Sigma_{n=2}^{\infty}\left(\frac{x}{\rho(\xi_0)}\right)^n\frac{1}{n}+C_0x}}\\&<\int_0^{\epsilon}dx\sqrt{\frac{2}{\left(\frac{1}{\rho(\xi_0)}+C_0\right)x}}\\&=\frac{2\sqrt{2}}{\sqrt{\left(\frac{1}{\rho(\xi_0)}+C_0\right)}}\sqrt{\epsilon},\end{aligned} \tag{23}$$

where we use Eq. (18). According to Eq. (23), $\lim_{\epsilon\to 0}Q_2<0$ for $1/\rho(\xi_0)+C_0>0$. If C_0 is positive, this condition does not cause any constraint. Even if C_0 is negative, this does not have any constraint because $\rho<1/|C_0|$ for this case, as we mentioned previously. Since Q_2 has an upper bound, Q must converge.

Let us show Eq. (13), which means $Q(A)$ is monotonically increasing as A becomes larger. By taking the derivative of both sides of Eq. (20),

$$\begin{aligned}\frac{1}{\sqrt{2}}\frac{dQ}{dA}&=\frac{1}{\sqrt{-\log\rho(\xi_0)-C_0\rho(\xi_0)+\log A}}\\&\quad+\int_0^{\rho(\xi_0)}d\rho\frac{\partial}{\partial A}\frac{1}{\sqrt{-\log\rho-C_0\rho+\log A}}.\end{aligned} \tag{24}$$

We have to regularize the first term by putting $\rho(\xi_0)-\epsilon$ instead of $\rho(\xi_0)$, because this is divergent. By replacing the derivative of A with that of ρ, the second term of the right-hand side is rewritten as

$$\begin{aligned}&\int_0^{\rho(\xi_0)}d\rho\frac{\partial}{\partial A}\frac{1}{\sqrt{-\log\rho-C_0\rho+\log A}}\\&\quad=\int_0^{\rho(\xi_0)}d\rho\frac{1}{-\frac{1}{\rho}-C_0}\frac{1}{A}\frac{\partial}{\partial\rho}\frac{1}{\sqrt{-\log\rho-C_0\rho+\log A}}.\end{aligned} \tag{25}$$

After we perform partial integration, we obtain

$$\frac{dQ}{dA}=\int_0^{\rho(\xi_0)}d\rho\frac{\sqrt{2}}{A}\frac{1}{(1+C_0\rho)^2}\frac{1}{\sqrt{-\log\rho-C_0\rho+\log A}}, \tag{26}$$

where we use $\partial\rho(\xi_0)/\partial A=-1/(A[-1/\rho(\xi_0)-C_0])$. Since Eq. (26) is positive definite, $Q(A)$ is a monotonically increasing function.

Finally, we show that $^{\exists}A,Q(A)>1$. With accuracy, $Q\to\infty$ as $A\to\infty$ for $C_0\geqslant 0$ and $Q\to\infty$ as $A\to 1/(|C_0|e)$ for $C_0<0$. First, we consider the case that C_0 is positive. If we consider the region where A is sufficiently large, the integral region in $Q(A)$ may be divided in the following way:

$$Q(A)=\left[\int_0^1+\int_1^{\rho(\xi_0)}\right]d\rho\sqrt{\frac{2}{-\log\rho-C_0\rho+\log A}}, \tag{27}$$

because $\rho(\xi_0)$ increases monotonically as A becomes larger from Eq. (18). The first term obviously goes to 0 as $A\to\infty$. For the second term,

$$\begin{aligned}&\int_1^{\rho(\xi_0)}d\rho\sqrt{\frac{2}{-\log\rho-C_0\rho+\log A}}\\&\quad>\int_1^{\rho(\xi_0)}d\rho\sqrt{\frac{2}{-C_0\rho+\log A}},\end{aligned} \tag{28}$$

where we use the fact that $\log\rho>0$ when $\rho>1$. The right-hand side goes to $2\sqrt{2}\sqrt{\log A}/C_0$ as $A\to\infty$. Since the lower bound of $Q(A)$ goes to infinity, $Q(A)$ must go to infinity. For the case $C_0<0$, the variable A has an upper bound of $1/(|C_0|e)$, which corresponds to $\rho(\xi_0)=1/|C_0|$. The charge $Q(1/(|C_0|e))$ is given by

$$Q\left(\frac{1}{(|C_0|e)}\right)=\int_0^{1/|C_0|}d\rho\ \sqrt{\frac{2}{-\log|C_0|\rho+|C_0|\rho-1}}. \tag{29}$$

Following the methods used before, we find that Eq. (29) is infinite.

We have shown that $Q(A)$ increases monotonically and that $Q(A)\to\infty$ as $A\to\infty$ for $C_0\geqslant 0$, $Q(A)\to\infty$ as $A\to 1/(|C_0|e)$ for $C_0<0$. Thus, it is proven analytically that the Haissinski equation with a capacitive wake function has a unique solution. Further, it has been proven analytically that the solution has only one relative maximum, and that it also has an upper bound, $\rho<1/|C_0|$, for the $C_0<0$ case. Although this wake function is a very special one, this work is meaningful in that we have shown how to prove the existence and uniqueness of the solution in an analytical manner. We analytically knew that the Haissinski equation had an unique solution for a resistive case because we obtained this solution explicitly. Here, we have found one example whose solution cannot be obtained explicitly, but by which we can analytically prove its solution's existence and uniqueness. This case causes us to hope that we will sometime prove the existence and uniqueness analytically only if we know the relation that the solution must satisfy.

It is of great interest to see whether the Haissinski equation has at least one solution for any physical wake function. Although it seems true empirically, we are far from its proof. The present paper is a step towards a solution to this problem. When we obtain a rigorous proof for the general wake functions, we will complete the basis of the theory of bunch lengthening.

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